

A HYPERDETERMINANT FOR $2 \times 2 \times 3$ ARRAYS

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ABSTRACT. We use the representation theory of Lie algebras and computational linear algebra to determine the simplest nonconstant invariant polynomial in the entries of a general $2 \times 2 \times 3$ array. This polynomial is homogeneous of degree 6 and has 66 terms with coefficients $\pm 1, \pm 2$ in the 12 indeterminates x_{ijk} where $i, j = 1, 2$ and $k = 1, 2, 3$. This invariant can be regarded as a natural generalization of Cayley's hyperdeterminant for $2 \times 2 \times 2$ arrays.

1. INTRODUCTION

A fundamental object in multilinear algebra is Cayley's hyperdeterminant [1], also called Kruskal's polynomial [9], a homogeneous polynomial of degree 4 in the 8 entries of a $2 \times 2 \times 2$ array. This polynomial plays an important role in the calculation of the rank of such an array; see ten Berge [13], and the recent papers by de Silva and Lim [3], Stegeman and Comon [11], and Martin [10]. For a comprehensive survey of the topic of tensor decomposition, see Kolda and Bader [8].

Gelfand, Kapranov and Zelevinsky [5] pointed out that Cayley's hyperdeterminant is the simplest (non-constant) polynomial in the entries of a $2 \times 2 \times 2$ array, regarded as an element of the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, which is invariant under the action of the Lie group $SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$. Inspired by this perspective, we use the representation theory of Lie algebras and computational linear algebra to determine the simplest (non-constant) polynomial invariant of the entries of a general $2 \times 2 \times 3$ array. This polynomial is homogeneous of degree 6 and has 66 terms with coefficients $\pm 1, \pm 2$.

In Section 2 we recall some basic definitions, and in Section 3 we present the details of our calculations, which were done with the computer algebra system Maple. The necessary background in Lie algebras and representation theory is summarized in Section 4.

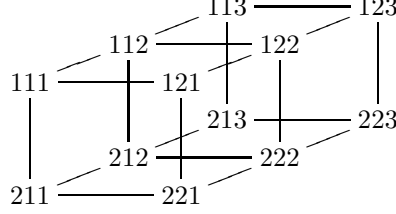
2. PRELIMINARIES

We consider a $2 \times 2 \times 3$ array $X = (x_{ijk})$ with $i, j \in \{1, 2\}$ and $k \in \{1, 2, 3\}$. We represent this array in terms of its three frontal slices:

$$X = \left[\begin{array}{cc|cc|cc} x_{111} & x_{121} & x_{112} & x_{122} & x_{113} & x_{123} \\ x_{211} & x_{221} & x_{212} & x_{222} & x_{213} & x_{223} \end{array} \right].$$

The research of the author was partially supported by a Discovery Grant from NSERC, the Natural Sciences and Engineering Research Council of Canada.

Geometrically, such an array can be represented by the following diagram:



A monomial in the entries of the array X has the form

$$M = x_{111}^{e_{111}} x_{121}^{e_{121}} x_{211}^{e_{211}} x_{221}^{e_{221}} x_{112}^{e_{112}} x_{122}^{e_{122}} x_{212}^{e_{212}} x_{222}^{e_{222}} x_{113}^{e_{113}} x_{123}^{e_{123}} x_{213}^{e_{213}} x_{223}^{e_{223}},$$

corresponding to a $2 \times 2 \times 3$ array $E = (e_{ijk})$ of non-negative integer exponents:

$$E = \left[\begin{array}{cc|cc|cc} e_{111} & e_{121} & e_{112} & e_{122} & e_{113} & e_{123} \\ e_{211} & e_{221} & e_{212} & e_{222} & e_{213} & e_{223} \end{array} \right].$$

The degree of a monomial M is the sum of its exponents:

$$e_{111} + e_{121} + e_{211} + e_{221} + e_{112} + e_{122} + e_{212} + e_{222} + e_{113} + e_{123} + e_{213} + e_{223}.$$

We define the weight of a monomial M to be the ordered quadruple of integers,

$$[w_1(M), w_2(M), w_{31}(M), w_{32}(M)],$$

where the components are defined as follows:

$$w_1(M) = e_{111} + e_{121} - e_{211} - e_{221} + e_{112} + e_{122} - e_{212} - e_{222} + e_{113} + e_{123} - e_{213} - e_{223},$$

$$w_2(M) = e_{111} - e_{121} + e_{211} - e_{221} + e_{112} - e_{122} + e_{212} - e_{222} + e_{113} - e_{123} + e_{213} - e_{223},$$

$$w_{31}(M) = e_{111} + e_{121} + e_{211} + e_{221} - e_{112} - e_{122} - e_{212} - e_{222},$$

$$w_{32}(M) = e_{112} + e_{122} + e_{212} + e_{222} - e_{113} - e_{123} - e_{213} - e_{223}.$$

The motivation for this definition of the weight comes from the representation theory of Lie algebras (see Section 4). We note that:

- $w_1(M)$ is the difference between the upper and lower horizontal slices;
- $w_2(M)$ is the difference between the left and right vertical slices;
- $w_{31}(M)$ is the difference between the first and second frontal slices;
- $w_{32}(M)$ is the difference between the second and third frontal slices.

We write P for the polynomial algebra generated by the entries of the array X over the field of complex numbers:

$$P = \mathbb{C}[x_{111}, x_{121}, x_{211}, x_{221}, x_{112}, x_{122}, x_{212}, x_{222}, x_{113}, x_{123}, x_{213}, x_{223}].$$

We have the direct sum decompositions

$$P = \bigoplus_{n \geq 0} P_n, \quad P_n = \bigoplus_{a, b, c_1, c_2 \in \mathbb{Z}} P_n(a, b, c_1, c_2),$$

where P_n is the subspace spanned by the monomials of degree n , and $P_n(a, b, c_1, c_2)$ is the subspace spanned by the monomials of weight $[a, b, c_1, c_2]$.

The representation theory of Lie algebras (see Section 4) shows that an invariant homogeneous polynomial of degree n belongs to $P_n(0, 0, 0, 0)$. A basis of this subspace consists of the monomials for which parallel slices in the exponent array E , in each of the three directions, have the same entry sum. It is clear that such

monomials exist if and only if n is a multiple of $\gcd(2, 2, 3) = 6$. We also consider four other subspaces,

$$P_n(2, 0, 0, 0), \quad P_n(0, 2, 0, 0), \quad P_n(0, 0, 2, -1), \quad P_n(0, 0, -1, 2).$$

We define four linear maps from $P_n(0, 0, 0, 0)$ to the other subspaces, again motivated by the representation theory of Lie algebras:

$$\begin{aligned} U_1: P_n(0, 0, 0, 0) &\rightarrow P_n(2, 0, 0, 0), & U_2: P_n(0, 0, 0, 0) &\rightarrow P_n(0, 2, 0, 0), \\ U_{31}: P_n(0, 0, 0, 0) &\rightarrow P_n(0, 0, 2, -1), & U_{32}: P_n(0, 0, 0, 0) &\rightarrow P_n(0, 0, -1, 2). \end{aligned}$$

These maps are defined on basis monomials and extended linearly. For U_1 , if $e_{2jk} \geq 1$ for some j, k then we multiply the monomial by e_{2jk} , subtract 1 from the exponent of x_{2jk} , and add 1 to the exponent of x_{1jk} ; the result of applying U_1 is the sum of these six terms (if $e_{2jk} = 0$ for some j, k then the corresponding term does not appear). For U_2 , the definition is similar, but we consider the second index: if $e_{i2k} \geq 1$ for some i, k then we multiply the monomial by e_{i2k} , subtract 1 from the exponent of x_{i2k} , and add 1 to the exponent of x_{i1k} . We have:

$$\begin{aligned} U_1 &\left(x_{111}^{e_{111}} x_{121}^{e_{121}} x_{211}^{e_{211}} x_{221}^{e_{221}} x_{112}^{e_{112}} x_{122}^{e_{122}} x_{212}^{e_{212}} x_{222}^{e_{222}} x_{113}^{e_{113}} x_{123}^{e_{123}} x_{213}^{e_{213}} x_{223}^{e_{223}} \right) = \\ &\quad e_{211} x_{111}^{e_{111}+1} x_{121}^{e_{121}} x_{211}^{e_{211}-1} x_{221}^{e_{221}} x_{112}^{e_{112}} x_{122}^{e_{122}} x_{212}^{e_{212}} x_{222}^{e_{222}} x_{113}^{e_{113}} x_{123}^{e_{123}} x_{213}^{e_{213}} x_{223}^{e_{223}} + \\ &\quad e_{221} x_{111}^{e_{111}} x_{121}^{e_{121}+1} x_{211}^{e_{211}} x_{221}^{e_{221}-1} x_{112}^{e_{112}} x_{122}^{e_{122}} x_{212}^{e_{212}} x_{222}^{e_{222}} x_{113}^{e_{113}} x_{123}^{e_{123}} x_{213}^{e_{213}} x_{223}^{e_{223}} + \\ &\quad e_{212} x_{111}^{e_{111}} x_{121}^{e_{121}} x_{211}^{e_{211}} x_{221}^{e_{221}} x_{112}^{e_{112}+1} x_{122}^{e_{122}} x_{212}^{e_{212}-1} x_{222}^{e_{222}} x_{113}^{e_{113}} x_{123}^{e_{123}} x_{213}^{e_{213}} x_{223}^{e_{223}} + \\ &\quad e_{222} x_{111}^{e_{111}} x_{121}^{e_{121}} x_{211}^{e_{211}} x_{221}^{e_{221}} x_{112}^{e_{112}} x_{122}^{e_{122}+1} x_{212}^{e_{212}} x_{222}^{e_{222}-1} x_{113}^{e_{113}} x_{123}^{e_{123}} x_{213}^{e_{213}} x_{223}^{e_{223}} + \\ &\quad e_{213} x_{111}^{e_{111}} x_{121}^{e_{121}} x_{211}^{e_{211}} x_{221}^{e_{221}} x_{112}^{e_{112}} x_{122}^{e_{122}} x_{212}^{e_{212}} x_{222}^{e_{222}} x_{113}^{e_{113}+1} x_{123}^{e_{123}} x_{213}^{e_{213}-1} x_{223}^{e_{223}} + \\ &\quad e_{223} x_{111}^{e_{111}} x_{121}^{e_{121}} x_{211}^{e_{211}} x_{221}^{e_{221}} x_{112}^{e_{112}} x_{122}^{e_{122}} x_{212}^{e_{212}} x_{222}^{e_{222}} x_{113}^{e_{113}} x_{123}^{e_{123}+1} x_{213}^{e_{213}} x_{223}^{e_{223}-1}, \\ U_2 &\left(x_{111}^{e_{111}} x_{121}^{e_{121}} x_{211}^{e_{211}} x_{221}^{e_{221}} x_{112}^{e_{112}} x_{122}^{e_{122}} x_{212}^{e_{212}} x_{222}^{e_{222}} x_{113}^{e_{113}} x_{123}^{e_{123}} x_{213}^{e_{213}} x_{223}^{e_{223}} \right) = \\ &\quad e_{121} x_{111}^{e_{111}+1} x_{121}^{e_{121}-1} x_{211}^{e_{211}} x_{221}^{e_{221}} x_{112}^{e_{112}} x_{122}^{e_{122}} x_{212}^{e_{212}} x_{222}^{e_{222}} x_{113}^{e_{113}} x_{123}^{e_{123}} x_{213}^{e_{213}} x_{223}^{e_{223}} + \\ &\quad e_{221} x_{111}^{e_{111}} x_{121}^{e_{121}} x_{211}^{e_{211}+1} x_{221}^{e_{221}-1} x_{112}^{e_{112}} x_{122}^{e_{122}} x_{212}^{e_{212}} x_{222}^{e_{222}} x_{113}^{e_{113}} x_{123}^{e_{123}} x_{213}^{e_{213}} x_{223}^{e_{223}} + \\ &\quad e_{122} x_{111}^{e_{111}} x_{121}^{e_{121}} x_{211}^{e_{211}} x_{221}^{e_{221}} x_{112}^{e_{112}+1} x_{122}^{e_{122}-1} x_{212}^{e_{212}} x_{222}^{e_{222}} x_{113}^{e_{113}} x_{123}^{e_{123}} x_{213}^{e_{213}} x_{223}^{e_{223}} + \\ &\quad e_{222} x_{111}^{e_{111}} x_{121}^{e_{121}} x_{211}^{e_{211}} x_{221}^{e_{221}} x_{112}^{e_{112}} x_{122}^{e_{122}} x_{212}^{e_{212}+1} x_{222}^{e_{222}-1} x_{113}^{e_{113}} x_{123}^{e_{123}} x_{213}^{e_{213}} x_{223}^{e_{223}} + \\ &\quad e_{123} x_{111}^{e_{111}} x_{121}^{e_{121}} x_{211}^{e_{211}} x_{221}^{e_{221}} x_{112}^{e_{112}} x_{122}^{e_{122}} x_{212}^{e_{212}} x_{222}^{e_{222}} x_{113}^{e_{113}+1} x_{123}^{e_{123}-1} x_{213}^{e_{213}} x_{223}^{e_{223}} + \\ &\quad e_{223} x_{111}^{e_{111}} x_{121}^{e_{121}} x_{211}^{e_{211}} x_{221}^{e_{221}} x_{112}^{e_{112}} x_{122}^{e_{122}} x_{212}^{e_{212}} x_{222}^{e_{222}} x_{113}^{e_{113}} x_{123}^{e_{123}} x_{213}^{e_{213}+1} x_{223}^{e_{223}-1}. \end{aligned}$$

For U_{31} , if $e_{ij2} \geq 1$ for some i, j then we multiply the monomial by e_{ij2} , subtract 1 from the exponent of x_{ij2} , and add 1 to the exponent of x_{ij1} . For U_{32} , if $e_{ij3} \geq 1$ for some i, j then we multiply the monomial by e_{ij3} , subtract 1 from the exponent of x_{ij3} , and add 1 to the exponent of x_{ij2} . We have:

$$\begin{aligned} U_{31} &\left(x_{111}^{e_{111}} x_{121}^{e_{121}} x_{211}^{e_{211}} x_{221}^{e_{221}} x_{112}^{e_{112}} x_{122}^{e_{122}} x_{212}^{e_{212}} x_{222}^{e_{222}} x_{113}^{e_{113}} x_{123}^{e_{123}} x_{213}^{e_{213}} x_{223}^{e_{223}} \right) = \\ &\quad e_{112} x_{111}^{e_{111}+1} x_{121}^{e_{121}} x_{211}^{e_{211}} x_{221}^{e_{221}} x_{112}^{e_{112}-1} x_{122}^{e_{122}} x_{212}^{e_{212}} x_{222}^{e_{222}} x_{113}^{e_{113}} x_{123}^{e_{123}} x_{213}^{e_{213}} x_{223}^{e_{223}} + \\ &\quad e_{122} x_{111}^{e_{111}} x_{121}^{e_{121}+1} x_{211}^{e_{211}} x_{221}^{e_{221}} x_{112}^{e_{112}} x_{122}^{e_{122}-1} x_{212}^{e_{212}} x_{222}^{e_{222}} x_{113}^{e_{113}} x_{123}^{e_{123}} x_{213}^{e_{213}} x_{223}^{e_{223}} + \\ &\quad e_{212} x_{111}^{e_{111}} x_{121}^{e_{121}} x_{211}^{e_{211}+1} x_{221}^{e_{221}} x_{112}^{e_{112}} x_{122}^{e_{122}} x_{212}^{e_{212}-1} x_{222}^{e_{222}} x_{113}^{e_{113}} x_{123}^{e_{123}} x_{213}^{e_{213}} x_{223}^{e_{223}} + \\ &\quad e_{222} x_{111}^{e_{111}} x_{121}^{e_{121}} x_{211}^{e_{211}} x_{221}^{e_{221}+1} x_{112}^{e_{112}} x_{122}^{e_{122}} x_{212}^{e_{212}} x_{222}^{e_{222}-1} x_{113}^{e_{113}} x_{123}^{e_{123}} x_{213}^{e_{213}} x_{223}^{e_{223}}, \end{aligned}$$

$$\begin{aligned}
U_{32} \Big(& x_{111}^{e_{111}} x_{121}^{e_{121}} x_{211}^{e_{211}} x_{221}^{e_{221}} x_{112}^{e_{112}} x_{122}^{e_{122}} x_{212}^{e_{212}} x_{222}^{e_{222}} x_{113}^{e_{113}} x_{123}^{e_{123}} x_{213}^{e_{213}} x_{223}^{e_{223}} \Big) = \\
& e_{113} x_{111}^{e_{111}} x_{121}^{e_{121}} x_{211}^{e_{211}} x_{221}^{e_{221}} x_{112}^{e_{112}+1} x_{122}^{e_{122}} x_{212}^{e_{212}} x_{222}^{e_{222}} x_{113}^{e_{113}-1} x_{123}^{e_{123}} x_{213}^{e_{213}} x_{223}^{e_{223}} + \\
& e_{123} x_{111}^{e_{111}} x_{121}^{e_{121}} x_{211}^{e_{211}} x_{221}^{e_{221}} x_{112}^{e_{112}} x_{122}^{e_{122}+1} x_{212}^{e_{212}} x_{222}^{e_{222}} x_{113}^{e_{113}} x_{123}^{e_{123}-1} x_{213}^{e_{213}} x_{223}^{e_{223}} + \\
& e_{213} x_{111}^{e_{111}} x_{121}^{e_{121}} x_{211}^{e_{211}} x_{221}^{e_{221}} x_{112}^{e_{112}} x_{122}^{e_{122}} x_{212}^{e_{212}+1} x_{222}^{e_{222}} x_{113}^{e_{113}} x_{123}^{e_{123}} x_{213}^{e_{213}-1} x_{223}^{e_{223}} + \\
& e_{223} x_{111}^{e_{111}} x_{121}^{e_{121}} x_{211}^{e_{211}} x_{221}^{e_{221}} x_{112}^{e_{112}} x_{122}^{e_{122}} x_{212}^{e_{212}} x_{222}^{e_{222}+1} x_{113}^{e_{113}} x_{123}^{e_{123}} x_{213}^{e_{213}} x_{223}^{e_{223}-1}.
\end{aligned}$$

We combine these four linear maps into a single map and consider

$$U: P_n(0, 0, 0, 0) \longrightarrow P_n(2, 0, 0, 0) \oplus P_n(0, 2, 0, 0) \oplus P_n(0, 0, 2, -1) \oplus P_n(0, 0, -1, 2),$$

defined on basis monomials by the equation

$$U(M) = (U_1(M), U_2(M), U_{31}(M), U_{32}(M)).$$

It follows from the representation theory of Lie algebras that the invariant polynomials of degree n are the elements of the kernel of U .

3. MAIN RESULT

Ignoring the trivial invariant in degree 0 — the constant polynomial 1 — the lowest degree in which an invariant polynomial can exist is 6. We adopt the convention of flattening an array of exponents as follows:

$$\left[\begin{array}{cc|cc|cc} e_{111} & e_{121} & e_{112} & e_{122} & e_{113} & e_{123} \\ e_{211} & e_{221} & e_{212} & e_{222} & e_{213} & e_{223} \end{array} \right] \longleftrightarrow \left[e_{111} \ e_{121} \ e_{211} \ e_{221} \ e_{112} \ e_{122} \ e_{212} \ e_{222} \ e_{113} \ e_{123} \ e_{213} \ e_{223} \right].$$

With this notation, the 80 basis monomials of $P_6(0, 0, 0, 0)$ are as follows, listed in lexicographical order:

200010010002	200001100002	200001010011	200000110101	200000021001
200000020110	110010100002	110010010011	110001100011	110001010020
110000200101	110000111001	110000110110	110000021010	101011000002
101010010101	101002000011	101001100101	101001011001	101001010110
101000110200	101000021100	100120000002	100111000011	100110100101
100110011001	100110010110	100102000020	100101101001	100101100110
100101011010	100100200200	100100111100	100100022000	020010100011
020010010020	020001100020	020000201001	020000200110	020000111010
011020000002	011011000011	011010100101	011010011001	011010010110
011002000020	011001101001	011001100110	011001011010	011000200200
011000111100	011000022000	010120000011	010111000020	010110101001
010110100110	010110011010	010101101010	010100201100	010100112000
002011000101	002010010200	002002001001	002002000110	002001100200
002001011100	001120000101	001111001001	001111000110	001110100200
001110011100	001102001010	001101101100	001101012000	000220001001
000220000110	000211001010	000210101100	000210012000	000201102000

We also consider the four subspaces,

$$P_6(2, 0, 0, 0), \quad P_6(0, 2, 0, 0), \quad P_6(0, 0, 2, -1), \quad P_6(0, 0, -1, 2),$$

with dimensions 63, 63, 60, 60 respectively.

In degree 6, we represent the linear map U as the matrix $[U]$ with respect to the lexicographically ordered bases of the five subspaces. The matrix $[U]$ has 80

columns and $63 + 63 + 60 + 60 = 246$ rows; it consists of a stack of four blocks, two of size 63×80 and two of size 60×80 . We use the computer algebra system Maple to construct the matrix $[U]$ and compute its row canonical form; we find that the rank is 79 and hence the nullspace is 1-dimensional. This provides a computational verification that there exist invariant polynomials in degree 6, and that every such polynomial is a scalar multiple of one fundamental invariant. The 1×80 coefficient vector of a nullspace basis can be represented by this 4×20 matrix:

$$\begin{array}{cccccccccccccccccccc} 0 & 1 & -1 & -1 & 0 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 0 & 1 & 1 & 0 & -2 & -1 & -2 & 2 & 1 & -1 & 1 & 0 & 1 & -1 & 0 & -1 & 0 & 1 \\ 1 & -1 & -1 & -2 & 2 & 0 & 2 & 0 & -1 & 0 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 0 & 1 & -1 & -1 & 0 & 1 \end{array}$$

The 66 nonzero coefficients are ± 1 and ± 2 . In each monomial, the exponents form a partition of 6, either 2211 or 21111 or 111111. We call this polynomial \mathcal{D} .

To understand the structure of the invariant polynomial \mathcal{D} , we consider the action of the group $S_2 \times S_2 \times S_3$, where S_n is the group of permutations of n symbols, on the indices (i, j, k) in the Cartesian product $\{1, 2\} \times \{1, 2\} \times \{1, 2, 3\}$ corresponding to the indeterminates x_{ijk} . Let α (respectively β) be the transposition (12) in the first (respectively second) copy of S_2 , which we denote by $S_{2,1}$ (respectively $S_{2,2}$). Let σ and τ be the transposition (12) and the 3-cycle (123) in S_3 . We have

$$S_{2,1} = \{1, \alpha\}, \quad S_{2,2} = \{1, \beta\}, \quad S_3 = \{1, \tau, \tau^2, \sigma, \sigma\tau, \sigma\tau^2\}.$$

Given a monomial M of degree 6 and weight $[0, 0, 0, 0]$, we consider the following element of $P_6(0, 0, 0, 0)$, which we call the signed orbit of M under the action of $S_{2,1} \times S_{2,2} \times S_3$ on the subscripts of its indeterminates:

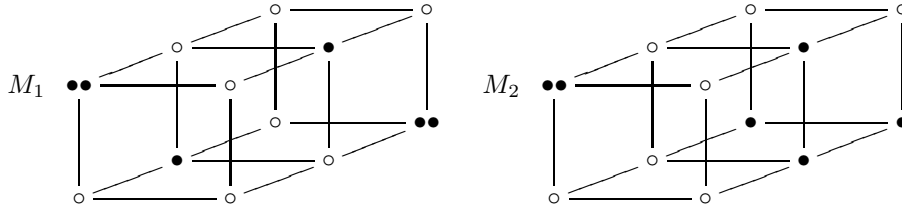
$$\begin{aligned} \text{orbit}(M) = & M + \tau M + \tau^2 + \sigma M + \sigma\tau M + \sigma\tau^2 M \\ & - \beta M - \beta\tau M - \beta\tau^2 - \beta\sigma M - \beta\sigma\tau M - \beta\sigma\tau^2 M \\ & - \alpha M - \alpha\tau M - \alpha\tau^2 - \alpha\sigma M - \alpha\sigma\tau M - \alpha\sigma\tau^2 M \\ & + \alpha\beta M + \alpha\beta\tau M + \alpha\beta\tau^2 + \alpha\beta\sigma M + \alpha\beta\sigma\tau M + \alpha\beta\sigma\tau^2 M. \end{aligned}$$

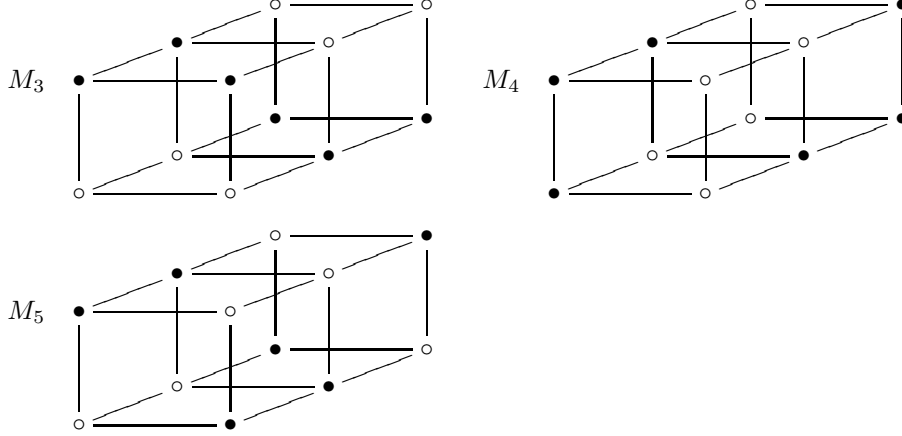
The sign of each term is the product of the signs of the corresponding elements of $S_{2,1}$ and $S_{2,2}$ (we ignore the sign of the element of S_3).

In particular, we consider the following five monomials:

$$\begin{aligned} M_1 &= x_{111}^2 x_{122} x_{212} x_{223}^2, & M_2 &= x_{111}^2 x_{122} x_{222} x_{213} x_{223}, \\ M_3 &= x_{111} x_{121} x_{112} x_{222} x_{213} x_{223}, & M_4 &= x_{111} x_{211} x_{112} x_{222} x_{123} x_{223}, \\ M_5 &= x_{111} x_{221} x_{112} x_{222} x_{123} x_{213}. \end{aligned}$$

Geometrically these monomials are represented by the following diagrams, where a solid (respectively open) vertex indicates that the corresponding indeterminate does (respectively does not) occur; a double solid vertex indicates the square:





The hyperdeterminant \mathcal{D} of a $2 \times 2 \times 3$ array is a linear combination of the orbits generated by these five monomials. A straightforward calculation verifies the following expressions. We note that the orbits of M_3 and M_4 are interchanged by the transposition of the first two indices, which corresponds to the standard matrix transposition of the 2×2 frontal slices:

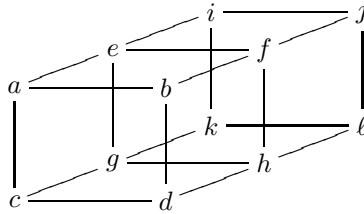
$$\begin{aligned}
& \frac{1}{2}\text{orbit}(M_1) = \\
& \quad x_{111}^2 x_{122} x_{212} x_{223}^2 + x_{111}^2 x_{222} x_{123} x_{213} - x_{111} x_{221} x_{122}^2 x_{213}^2 \\
& \quad - x_{111} x_{221} x_{212}^2 x_{123}^2 - x_{121}^2 x_{112} x_{222} x_{213}^2 - x_{121}^2 x_{212}^2 x_{113} x_{223} \\
& \quad + x_{121} x_{211} x_{112}^2 x_{223}^2 + x_{121} x_{211} x_{222}^2 x_{113}^2 - x_{211}^2 x_{112} x_{222} x_{123}^2 \\
& \quad - x_{211}^2 x_{122}^2 x_{113} x_{223} + x_{221}^2 x_{112}^2 x_{123} x_{213} + x_{221}^2 x_{122} x_{212} x_{113}^2, \\
& - \text{orbit}(M_2) = \\
& \quad - x_{111}^2 x_{122} x_{222} x_{213} x_{223} - x_{111}^2 x_{212} x_{222} x_{123} x_{223} - x_{111} x_{121} x_{112} x_{212} x_{223}^2 \\
& \quad + x_{111} x_{121} x_{122} x_{222} x_{213}^2 + x_{111} x_{121} x_{212}^2 x_{123} x_{223} - x_{111} x_{121} x_{222}^2 x_{113} x_{213} \\
& \quad - x_{111} x_{211} x_{112} x_{122} x_{223}^2 + x_{111} x_{211} x_{122}^2 x_{213} x_{223} + x_{111} x_{211} x_{212} x_{222} x_{123}^2 \\
& \quad - x_{111} x_{211} x_{222}^2 x_{113} x_{123} + x_{121}^2 x_{112} x_{212} x_{213} x_{223} + x_{121}^2 x_{212} x_{222} x_{113} x_{213} \\
& \quad - x_{121} x_{221} x_{112}^2 x_{213} x_{223} + x_{121} x_{221} x_{112} x_{122} x_{213}^2 + x_{121} x_{221} x_{212}^2 x_{113} x_{123} \\
& \quad - x_{121} x_{221} x_{212} x_{222} x_{113}^2 + x_{211}^2 x_{112} x_{122} x_{123} x_{223} + x_{211}^2 x_{122} x_{222} x_{113} x_{123} \\
& \quad - x_{211} x_{221} x_{112}^2 x_{123} x_{223} + x_{211} x_{221} x_{112} x_{212} x_{123}^2 + x_{211} x_{221} x_{122}^2 x_{113} x_{213} \\
& \quad - x_{211} x_{221} x_{122} x_{222} x_{113}^2 - x_{221}^2 x_{112} x_{122} x_{113} x_{213} - x_{221}^2 x_{112} x_{212} x_{113} x_{123}, \\
& \frac{1}{2}\text{orbit}(M_3) = \\
& \quad x_{111} x_{121} x_{112} x_{222} x_{213} x_{223} - x_{111} x_{121} x_{122} x_{212} x_{213} x_{223} \\
& \quad + x_{111} x_{121} x_{212} x_{222} x_{113} x_{223} - x_{111} x_{121} x_{212} x_{222} x_{123} x_{213} \\
& \quad + x_{111} x_{221} x_{112} x_{122} x_{213} x_{223} + x_{111} x_{221} x_{212} x_{222} x_{113} x_{123} \\
& \quad - x_{121} x_{211} x_{112} x_{122} x_{213} x_{223} - x_{121} x_{211} x_{212} x_{222} x_{113} x_{123} \\
& \quad + x_{211} x_{221} x_{112} x_{122} x_{113} x_{223} - x_{211} x_{221} x_{112} x_{122} x_{123} x_{213} \\
& \quad + x_{211} x_{221} x_{112} x_{222} x_{113} x_{123} - x_{211} x_{221} x_{122} x_{212} x_{113} x_{123},
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2}\text{orbit}(M_4) = \\
& x_{111}x_{211}x_{112}x_{222}x_{123}x_{223} - x_{111}x_{211}x_{122}x_{212}x_{123}x_{223} \\
& + x_{111}x_{211}x_{122}x_{222}x_{113}x_{223} - x_{111}x_{211}x_{122}x_{222}x_{123}x_{213} \\
& + x_{111}x_{221}x_{112}x_{212}x_{123}x_{223} + x_{111}x_{221}x_{122}x_{222}x_{113}x_{213} \\
& - x_{121}x_{211}x_{112}x_{212}x_{123}x_{223} - x_{121}x_{211}x_{122}x_{222}x_{113}x_{213} \\
& + x_{121}x_{221}x_{112}x_{212}x_{113}x_{223} - x_{121}x_{221}x_{112}x_{212}x_{123}x_{213} \\
& + x_{121}x_{221}x_{112}x_{222}x_{113}x_{213} - x_{121}x_{221}x_{122}x_{212}x_{113}x_{213}, \\
& - \frac{1}{2}\text{orbit}(M_5) = \\
& - 2x_{111}x_{221}x_{112}x_{222}x_{123}x_{213} - 2x_{111}x_{221}x_{122}x_{212}x_{113}x_{223} \\
& + 2x_{111}x_{221}x_{122}x_{212}x_{123}x_{213} - 2x_{121}x_{211}x_{112}x_{222}x_{113}x_{223} \\
& + 2x_{121}x_{211}x_{112}x_{222}x_{123}x_{213} + 2x_{121}x_{211}x_{122}x_{212}x_{113}x_{223}.
\end{aligned}$$

Theorem. *The simplest (non-constant) invariant polynomial for $2 \times 2 \times 3$ arrays is a homogeneous polynomial \mathcal{D} of degree 6 with 66 terms and coefficients ± 1 and ± 2 . It is equal to the following linear combination of the signed orbits of five monomials under the action of the group $S_2 \times S_2 \times S_3$:*

$$\begin{aligned}
\mathcal{D} = & \frac{1}{2}\text{orbit}(x_{111}^2x_{122}x_{212}x_{223}^2) - \text{orbit}(x_{111}^2x_{122}x_{222}x_{213}x_{223}) \\
& + \frac{1}{2}\text{orbit}(x_{111}x_{121}x_{112}x_{222}x_{213}x_{223}) + \frac{1}{2}\text{orbit}(x_{111}x_{211}x_{112}x_{222}x_{123}x_{223}) \\
& - \frac{1}{2}\text{orbit}(x_{111}x_{221}x_{112}x_{222}x_{123}x_{213}).
\end{aligned}$$

We can express the same result, avoiding subscripts, as follows. The hyperdeterminant \mathcal{D} of the $2 \times 2 \times 3$ array



is the polynomial

$$\begin{aligned}
\mathcal{D} = & a^2fg\ell^2 + a^2h^2jk - adf^2k^2 - adg^2j^2 - b^2ehk^2 - b^2g^2il \\
& + bce^2\ell^2 + bch^2i^2 - c^2ehj^2 - c^2f^2il + d^2e^2jk + d^2fgi^2 \\
& - a^2fhkl - a^2ghjl - abeg\ell^2 + abfhk^2 + abg^2jl - abh^2ik \\
& - acef\ell^2 + acf^2kl + acghj^2 - ach^2ij + b^2egkl + b^2ghik \\
& - bde^2kl + bdefk^2 + bdg^2ij - bdghi^2 + c^2efjl + c^2fhi^2 \\
& - cde^2jl + cdegj^2 + cdf^2ik - cdfhi^2 - d^2efik - d^2egij \\
& + abehkl - abfgkl + abghil - abghjk + adfkl + adghi^2 \\
& - bcefk\ell - bcghij + cdefil - cdefjk + cdehij - cdfgij \\
& + acehjl - acfgjl + acfhi\ell - acfhjk + adegjl + adfhik \\
& - bcegg\ell - bcfhik + bdegil - bdegjk + bdehik - bdfgik \\
& - 2adehjk - 2adfgil + 2adfgjk - 2bcehil + 2bcehjk + 2bcfgil.
\end{aligned}$$

4. REPRESENTATION THEORY OF LIE ALGEBRAS

This section provides a brief summary of the necessary background material on Lie algebras and their representations. Details may be found in standard textbooks such as Jacobson [7], Humphreys [6], de Graaf [2], or Erdmann and Wildon [4].

We regard a $2 \times 2 \times 3$ array with complex entries as an element of the tensor product $T = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$. The 14-dimensional semisimple Lie group

$$G = SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times SL_3(\mathbb{C}),$$

acts on the 12-dimensional space T by unimodular changes of basis along the three directions. ($SL_n(\mathbb{C})$ is the set of $n \times n$ complex matrices with determinant 1, with the usual definition of matrix multiplication.) Since we are concerned only with the action of this Lie group on finite-dimensional complex vector spaces, we can linearize the problem and consider instead the action of the Lie algebra

$$L = sl_2(\mathbb{C}) \oplus sl_2(\mathbb{C}) \oplus sl_3(\mathbb{C}).$$

($sl_n(\mathbb{C})$ is the vector space of $n \times n$ complex matrices with trace 0; two such matrices are composed using the commutator $[A, B] = AB - BA$.) For an elementary and attractive introduction to Lie theory, by which is meant the connection between Lie groups and Lie algebras, see Stillwell [12].

The most important elements of $sl_2(\mathbb{C})$ and $sl_3(\mathbb{C})$ are the diagonal matrices,

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

and the superdiagonal matrices,

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

In the natural representation, the elements of $sl_2(\mathbb{C})$ and $sl_3(\mathbb{C})$ act by left matrix-vector multiplication on the vector spaces \mathbb{C}^2 and \mathbb{C}^3 with standard bases,

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(The context will clarify this ambiguous notation.) From this we obtain the basis of T consisting of the simple tensors

$$x_{ijk} = x_i \otimes x_j \otimes x_k \quad (i, j \in \{1, 2\}, k \in \{1, 2, 3\}).$$

Strictly speaking, we regard this element as a coordinate function on T , so we should use dual basis vectors, but this distinction will not matter for us.

We need to determine the action of the basis of L on the basis of T . We have

$$\begin{aligned} H \cdot x_1 &= x_1, & H \cdot x_2 &= -x_2, \\ E \cdot x_1 &= 0, & E \cdot x_2 &= x_1, \\ H_1 \cdot x_1 &= x_1, & H_1 \cdot x_2 &= -x_2, & H_1 \cdot x_3 &= 0, \\ H_2 \cdot x_1 &= 0, & H_2 \cdot x_2 &= x_2, & H_2 \cdot x_3 &= -x_3, \\ E_1 \cdot x_1 &= 0, & E_1 \cdot x_2 &= x_1, & E_1 \cdot x_3 &= 0, \\ E_2 \cdot x_1 &= 0, & E_2 \cdot x_2 &= 0, & E_2 \cdot x_3 &= x_2. \end{aligned}$$

The general element (A, B, C) in L acts on simple tensors in T as follows:

$$\begin{aligned} (A, B, C) \cdot (x_i \otimes x_j \otimes x_k) \\ = (A \cdot x_i) \otimes x_j \otimes x_k + x_i \otimes (B \cdot x_j) \otimes x_k + x_i \otimes x_j \otimes (C \cdot x_k). \end{aligned}$$

This action extends to the monomial basis of the polynomial algebra

$$P = \mathbb{C}[x_{111}, x_{121}, x_{211}, x_{221}, x_{112}, x_{122}, x_{212}, x_{222}, x_{113}, x_{123}, x_{213}, x_{223}],$$

by induction on the degree using the derivation rule (which generalizes the product rule from elementary calculus),

$$(A, B, C) \cdot (fg) = ((A, B, C) \cdot f)g + f((A, B, C) \cdot g);$$

the action then extends linearly to P . In particular, we obtain the following formulas (which generalize the extended power rule from elementary calculus):

$$\begin{aligned} H \cdot x_1^e &= ex_1^e, & H \cdot x_2^e &= -ex_2^e, \\ E \cdot x_1^e &= 0, & E \cdot x_2^e &= ex_1x_2^{e-1}, \\ H_1 \cdot x_1^e &= ex_1^e, & H_1 \cdot x_2^e &= -ex_2^e, & H_1 \cdot x_3^e &= 0, \\ H_2 \cdot x_1^e &= 0, & H_2 \cdot x_2^e &= ex_2^e, & H_2 \cdot x_3^e &= -ex_3^e, \\ E_1 \cdot x_1^e &= 0, & E_1 \cdot x_2^e &= ex_1x_2^{e-1}, & E_1 \cdot x_3^e &= 0, \\ E_2 \cdot x_1^e &= 0, & E_2 \cdot x_2^e &= 0, & E_2 \cdot x_3^e &= ex_2x_3^{e-1}. \end{aligned}$$

Lie theory shows that the polynomials which are fixed by the action of the Lie group G coincide with the polynomials which are annihilated by the action of the Lie algebra L . Furthermore, the representation theory of Lie algebras shows that a polynomial is annihilated by L if and only if it is annihilated by the elements

$$\begin{aligned} (H, 0, 0), \quad (E, 0, 0), \quad (0, H, 0), \quad (0, E, 0), \\ (0, 0, H_1), \quad (0, 0, H_2), \quad (0, 0, E_1), \quad (0, 0, E_2). \end{aligned}$$

Combining the previous formulas to determine the action on a general monomial

$$M = x_{111}^{e_{111}} x_{121}^{e_{121}} x_{211}^{e_{211}} x_{221}^{e_{221}} x_{112}^{e_{112}} x_{122}^{e_{122}} x_{212}^{e_{212}} x_{222}^{e_{222}} x_{113}^{e_{113}} x_{123}^{e_{123}} x_{213}^{e_{213}} x_{223}^{e_{223}},$$

we obtain

$$\begin{aligned} (H, 0, 0) \cdot M &= w_1(M), & (0, H, 0) \cdot M &= w_2(M), \\ (0, 0, H_1) \cdot M &= w_{31}(M), & (0, 0, H_2) \cdot M &= w_{32}(M), \end{aligned}$$

where $w_1(M)$, $w_2(M)$, $w_{31}(M)$, $w_{32}(M)$ are defined in Section 2. Thus a homogeneous polynomial of degree n is annihilated by the diagonal matrices H, H_1, H_2 in L if and only if it belongs to the subspace $P_n(0, 0, 0, 0)$. To conclude this summary of the representation theory, we observe that the action of the superdiagonal matrices E, E_1, E_2 in L is given by the linear maps U_1, U_2, U_{31}, U_{32} .

The dimension of the subspace $P_n(a, b, c_1, c_2)$ is of combinatorial interest, since a basis of this subspace consists of the $2 \times 2 \times 3$ arrays of non-negative integers with prescribed differences between the parallel slices in the three directions. Computational enumeration with Maple produced the dimensions in Table 1. Polynomial interpolation from the 17 data points in each column of Table 1 suggests the following conjecture for the dimensions of the weight spaces.

n	$\dim P_n(0, 0, 0, 0)$	$\dim P_n(2, 0, 0, 0)$ $\dim P_n(0, 2, 0, 0)$	$\dim P_n(0, 0, 2, -1)$ $\dim P_n(0, 0, -1, 2)$
0	1	0	0
6	80	63	60
12	1323	1206	1180
18	9832	9354	9240
24	46733	45294	44940
30	167184	163629	162740
36	491383	483732	481800
42	1250576	1235700	1231920
48	2851065	2824308	2817480
54	5959216	5913963	5902380
60	11610467	11537658	11518980
66	21345336	21232926	21204040
72	37375429	37207794	37164660
78	62782448	62539737	62477220
84	101753199	101410632	101322320
90	159853600	159380712	159258720
96	244344689	243704520	243539280

TABLE 1. Dimensions of weight spaces in degree n

Conjecture. *We have the following dimension formulas:*

$$\begin{aligned}
\dim P_n(0, 0, 0, 0) &= \frac{1}{58786560} (n+6) \times \\
&\quad (125n^6 + 4500n^5 + 68004n^4 + 552096n^3 + 2584224n^2 + 6811776n + 9797760), \\
\dim P_n(2, 0, 0, 0) &= \dim P_n(0, 2, 0, 0) = \frac{1}{58786560} n(n+6)(n+12) \times \\
&\quad (125n^4 + 3000n^3 + 28602n^2 + 127224n + 254664), \\
\dim P_n(0, 0, 2, -1) &= \dim P_n(0, 0, -1, 2) = \frac{1}{11757312} n(n+6)(n+12) \times \\
&\quad (5n^2 + 84n + 396)(5n^2 + 36n + 108)
\end{aligned}$$

In each case the function is a polynomial of degree 7.

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